

APPROXIMABLE ALGEBRAS AND A QUESTION OF H. CHEN.

CATRIONA MACLEAN
INSTITUT FOURIER
UNIVERSITÉ GRENoble ALPES.

ABSTRACT. In [2], Huayi Chen introduces the notion of an approximable graded algebra, and asks if any such algebra is a sub-algebra of the graded section ring of a big line bundle on an algebraic variety. We give a counter-example showing that this is not the case.

1. INTRODUCTION

The Fujita approximation theorem, [4], is an important result in algebraic geometry. It states that whilst the section ring associated to a big line bundle L on an algebraic variety X

$$R(L) \stackrel{\text{def}}{=} \bigoplus_n H^0(nL, X)$$

is typically not a finitely generated algebra, it can be approximated arbitrarily well by finitely generated algebras. More precisely, we have that

Theorem 1.1 (Fujita). *Let X be an algebraic variety and let L be a big line bundle on X . For any $\epsilon > 0$ there exists a birational modification*

$$\pi : \hat{X} \rightarrow X$$

and a decomposition of \mathbb{Q} divisors, $\pi^(L) = A + E$ such that*

- *A is ample and E is effective,*
- *$\text{vol}(A) \geq (1 - \epsilon)\text{vol}(L)$.*

In [6], Lazarsfeld and Mustata used the Newton-Okounkov body associated to A to give a simple proof of Fujita approximation. The Newton-Okounkov body, constructed in [5] and [6], building on previous work of Okounkov [7], is a convex body $\Delta_{Y_\bullet}(L, X)$ in \mathbb{R}^d associated to the data of

- a d -dimensional variety X
- an admissible flag Y_\bullet on X
- a big line bundle L on X .

This convex body encodes information on the asymptotic behaviour of the spaces of global sections $H^0(nL)$ for large values of L .

Lazarsfeld and Mustata's simple proof of Fujita approximation is based on the equality of volumes of Newton-Okounkov bodies

$$(1) \quad \text{vol}(L) = d! \text{vol}(\Delta_{Y_\bullet}(L, X))$$

where we recall that the volume of a big line bundle on a d -dimensional variety is defined by

$$\text{vol}(L) = \lim_{n \rightarrow \infty} \frac{d! h^0(nL)}{n^d}.$$

One advantage of their approach to the Fujita theorem is that Newton-Okounkov bodies are not only defined for section algebras $R(L)$, but also for any graded sub-algebra of section algebras. Lazarsfeld and Mustata give combinatorial conditions (conditions 2.3-2.5 of [6]) under which equation 1 holds for a graded sub-algebra $\oplus_n B_n \subset R(L)$ and show that these conditions hold if the subalgebra $\oplus_n B_n$ contains an ample series.¹

Di Biagio and Pacenzia in [3] subsequently used Newton-Okounkov bodies associated to restricted algebras to prove a Fujita approximation theorem for restricted linear series, ie. subalgebras of $\oplus_n H^0(nL|_V, V)$ obtained as the restriction of the complete algebra $\oplus_n H^0(nL, X)$, where $V \subset X$ is a subvariety.

In [2], Huayi Chen uses Lazarsfeld and Mustata's work on Fujita approximation to prove a Fujita-type approximation theorem in the arithmetic setting. In the course of this work he defines the notion of approximable graded algebras, which are exactly those algebras for which a Fujita-type approximation theorem hold.

Definition 1. *An integral graded algebra $B = \oplus_n B_n$ is approximable if and only if the following conditions are satisfied.*

- (1) *all the graded pieces B_n are finite dimensional.*
- (2) *for all sufficiently large n the space B_n is non-empty*
- (3) *for any ϵ there exists an p_0 such that for all $p \geq p_0$ we have that*

$$\liminf_{n \rightarrow \infty} \frac{\dim(\text{Im}(S^n B_p \rightarrow B_{np}))}{\dim(B_{np})} > (1 - \epsilon).$$

In his paper [2] Chen asks the following question.

Question 1 (Chen). *Let $B = \oplus_n B_n$ be an approximable graded algebra. Does there always exist a variety X and a big line bundle L such that B can be included as a subalgebra of $R(L)$?*

We will say that an approximable algebra for which the answer is “no” is *non-sectional*.

¹ie. if there exists an ample divisor $A \leq L$ such that $\oplus_n H^0([nA]) \subset B$

The aim of this note is to prove the following theorem

Theorem 1.2. *There exists an approximable graded algebra $B = \oplus_n B_n$ which is non-sectional.*

The basic criterion used will be the following : if B is an approximable graded algebra, we can consider the associated field of homogeneous quotients $K^{\text{gr}}(B)$, into which, after choice of a base element $b \in B_1$, the algebra B can be included.

If B is a graded sub-algebra of $\oplus_n H^0(nL)$ for some big line bundle L then the number of valuations of $K^{\text{gr}}(B)$ which take negative values on certain elements of B is finite.

In section 2) below we will state and prove our criterion for non-sectionality, whereas in section 3) we construct the non-sectional approximable algebra.

Note finally that while our counter example is not associated to a line bundle, it is of the form

$$\oplus_n H^0(\lfloor nD \rfloor)$$

for a certain infinite divisor $D = \sum_{i=0}^{\infty} a_i D_i$ with the property that $\lfloor nD \rfloor$ is finite for any choice of n . This poses the following natural question.

Question 2. *Let $B = \oplus_n B_n$ be a graded approximable algebra. Does there necessarily exist a variety X and a countable formal sum*

$$D = \oplus_i a_i D_i$$

of divisors on X such that for all n the integral divisor $\lfloor nD \rfloor$ is finite and B is isomorphic to the graded algebra $\oplus_n \mathcal{O}(\lfloor nD \rfloor)$?

2. A CRITERION FOR NOT-SECTIONALITY.

In this section, we will give a criterion for non-sectionality of a approximable graded algebra in terms of valuations on the associated field of homogeneous degree-zero rational functions.

Definition 2. *Let $B = \oplus_n B_n$ be an integral graded algebra. We define its graded field of functions by*

$$K^{\text{gr}}(B) = \left\{ \frac{f}{g} \mid \exists n \text{ s.t. } f, g \in B_n \right\} / \sim$$

where $\frac{f_1}{g_1} \sim \frac{f_2}{g_2}$ iff $f_1 g_2 = f_2 g_1$.

We now state our criterion.

Proposition 1. *Let $\oplus_n B_n$ be an approximable graded algebra with $B_0 = \mathbb{C}$. Suppose that the graded field of functions $K^{\text{gr}}(B)$ is a finitely generated field extension of \mathbb{C} with transcendence degree one and there is a choice of element $b_1 \in B_1$ such that the set*

$$\nu_{B,b_1} = \left\{ \nu \text{ valuation on } K^{\text{gr}}(B) \mid \exists n \in \mathbb{N}, b' \in B_n \text{ such that } \nu_B \left(\frac{b'}{b_1^n} \right) < 0 \right\}$$

is infinite. Then there does not exist a big line bundle L on a variety X such that $\oplus_n B_n$ is isomorphic as a graded algebra to a graded subalgebra of $\oplus_n H^0(nL, X)$.

Proof. Suppose that on the contrary we have a big line bundle L on a complex variety X and a graded inclusion $i : B \rightarrow R(L)$. This gives rise to a field inclusion

$$i^{\text{gr}} : K^{\text{gr}}(B) \rightarrow K^{\text{gr}}(L) = K(X).$$

After blowing up, we obtain a birational morphism $\pi^{\text{sm}} : X^{\text{sm}} \rightarrow X$ from a smooth variety X^{sm} . Consider the pull-back $\pi^{\text{sm}*}(L)$: there is an injective pull-back morphism $\pi^{\text{sm}*} : R(L) \rightarrow R(\pi^*(L))$. Replacing X with X^{sm} and i^{gr} with $\pi^* \circ i^{\text{gr}}$, we may assume that the variety X is smooth.

We have a distinguished element $b_1 \in B_1$, and an associated element $\sigma_1 = i^{\text{gr}}(b_1) \in H^0(L)$, so there are induced injective morphisms

$$\phi_B : B \hookrightarrow K^{\text{gr}}(B)$$

$$\phi_L : R(L) \hookrightarrow K^{\text{gr}}(L) = K(X)$$

given by $\phi_B(b_n) = \frac{b_n}{b_1^n}$ for all $b_n \in B_n$ and $\phi_L(\sigma_n) = \frac{\sigma_n}{(\sigma_1)^n}$ for all $\sigma_n \in H^0(nL)$. There is also an induced inclusion

$$i^{\text{gr}} : K^{\text{gr}}(B) \rightarrow K^{\text{gr}}(X)$$

given by $i^{\text{gr}} \left(\frac{f}{g} \right) = \frac{i(f)}{i(g)}$. By assumption, $K^{\text{gr}}(B)$ is finitely generated as a field extension of \mathbb{C} and is of transcendence degree 1 so there is a unique smooth complex curve C such that $K^{\text{gr}}(B) \sim K(C)$. After fixing an isomorphism $I : K(C) \rightarrow K^{\text{gr}}(B)$ we have an inclusion

$$K(C) \xrightarrow{i^{\text{gr}} \circ I} K(X)$$

and an induced rational morphism with dense image

$$\pi_X : X \dashrightarrow C$$

such that $i^{\text{gr}} \circ I = \pi_X^*$. After birational modification, we may assume that the map π_X is in fact a surjective morphism.

Valuations of $K(C)$ correspond to points of the curve C . Consider the set of valuations in ν_{B,b_1} , which by hypothesis is infinite

$$\nu_{B,b_1} = \{\nu_1, \nu_2, \dots, \nu_m, \dots\}$$

and consider the set of associated points in C ,

$$\{p_1, p_2, \dots, p_m, \dots\}$$

having the property that $\nu_j \circ I = \nu_{p_j}$. For any j we know that there exists a positive integer $n(j)$ and an element $b_j \in B_{n(j)}$ for some n such that

$$\nu_j \left(\frac{b_j}{b_1^{n(j)}} \right) < 0.$$

There exists a meromorphic function on C , f , such that $\frac{b_j}{b_1^{n(j)}} = I(f)$. Consider $i^{\text{gr}}(\frac{b_n}{b^n}) \in K(X)$: we have that

$$i^{\text{gr}} \left(\frac{b_n}{b^n} \right) = i^{\text{gr}}(I(f)) = \pi_X^*(f)$$

and hence

$$\pi_X^*(f) = \frac{i^{\text{gr}}(b_j)}{i^{\text{gr}}(b_1)^{n(j)}}.$$

We know that $\nu_{p_j}(f) = (\nu_j \circ I(f)) < 0$ or in other words f has a pole at the point p_j .

It follows that

$$i^{\text{gr}} \left(\frac{(b_j)}{(b_1)^{n(j)}} \right) = \frac{i(b_j)}{\sigma_1^{n(j)}}$$

has a pole along $\pi_X^{-1}(p_j)$ which is only possible if σ_1 has a zero along the divisor $\pi_X^{-1}(p_j)$. But this can only be the case for a finite number of points p_j .

This completes the proof of the Proposition 1. □

3. CONSTRUCTION OF THE EXAMPLE.

We will now construct our example of an approximable algebra $B = \bigoplus_n B_n$ such which satisfies the condition in Lemma 1.

We will start by associating to any number n an element of $\mathbb{N}^{\mathbb{N}}$.

Definition 3. Let n be a natural number. We denote by $I(n)$ the sequence

$$I(n) = \left(n, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{8} \rfloor, \dots \right).$$

We further denote by $J(n)$ the sum of the elements of $I(n)$, ie.

$$J(n) = n + \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{4} \rfloor + \lfloor \frac{n}{8} \rfloor + \dots$$

We will require the following superadditivity property of $I(n)$.

Lemma 1. *For all integers n and m we have that*

$$I(n) + I(m) \leq I(n + m)$$

where we consider that $v \leq w$ if $v_i \leq w_i$ for every integer

Proof. Immediate from the elementary fact that $\lfloor r + s \rfloor \geq \lfloor r \rfloor + \lfloor s \rfloor$ for any real numbers r and s . \square

Note that in particular $J(n) + J(m) \leq J(n + m)$ for all integers n and m .

We now choose an infinite sequence of distinct points in \mathbb{C} which we denote by z_0, \dots, z_m, \dots . Let x be a variable and consider the vector

$$\mathbf{a} = ((x - z_0), (x - z_1), \dots).$$

We associate to any integer n the polynomial in x

$$P_n(x) = \mathbf{a}^{I(n)} = \prod_{i=0}^{\infty} (x - z_i)^{\lfloor \frac{n}{2^i} \rfloor}$$

and we are now in a position to define our algebra.

Definition 4. *We define B_n by $B_n \subset \mathbb{C}(x)$,*

$$B_n = \left\{ \frac{Q(x)}{P_n(x)} \mid Q(x) \text{ polynomial of degree } J(n) \right\}.$$

The algebra B is then the graded algebra $B = \oplus_n B_n$

We note that $\oplus_n B_n$ is an algebra because for all n and m we have that $I(n) + I(m) \leq I(n + m)$ so that $P_n \times P_m \mid P_{n+m}$.

Remark 1. Note that while each of the subsets B_n is defined as a subset of $\mathbb{C}(x)$, the global algebra $\oplus_n B_n$ is not, since the subsets $B_n, B_m \subset \mathbb{C}(x)$ are not typically disjoint.

We now show that B is approximable.

Proposition 2. *The algebra $B = \oplus_n B_n$ is approximable.*

Proof. The conditions (1) and (2) of approximability are immediate. We turn to the proof of condition (3).

Note that the image in B_{pn} of $\text{Sym}^n(B_p)$ is given by

$$\frac{\mathbb{C}_{nJ(p)}[x]}{P_p(x)^n}$$

which is of dimension $nJ(p) + 1$ and that B_{np} is itself

$$\frac{\mathbb{C}_{J(pn)}[x]}{P_{pn}}$$

which is of dimension $J(pn) + 1$. We consider therefore the ratio

$$\frac{nJ(p) + 1}{J(pn) + 1}$$

and we will show that for sufficiently large values of p this ratio can be made arbitrarily close to 1.²

Note first that by the super-additivity property of J we have that for all p and n

$$nJ(p) \leq J(pn).$$

Moreover for all n we have that $J(n) \leq 2n$. Set $m = \lfloor \log_2(p) \rfloor$. We have that

$$\begin{aligned} 2p - J(p) &= \sum_{k=0}^{\infty} \left(\frac{p}{2^k} - \lfloor \frac{p}{2^k} \rfloor \right) \\ &= \sum_{k=0}^m \left(\frac{p}{2^k} - \lfloor \frac{p}{2^k} \rfloor \right) + \sum_{k=m+1}^{\infty} \left(\frac{p}{2^k} - \lfloor \frac{p}{2^k} \rfloor \right) \\ &\leq \sum_{k=1}^m 1 + \sum_{k=m+1}^{\infty} \frac{p}{2^{k+1}} \\ &= m + \frac{p}{2^m} \\ &\leq \log_2(p) + 2 \frac{p}{\log_2(p)}. \end{aligned}$$

In particular, $\lim_{p \rightarrow \infty} \frac{2p - J(p)}{p} = 0$. Suppose that p_0 such that for all $p \geq p_0$ we have that $2 - \frac{J(p)}{p} \leq \epsilon$ so that for all n and all $p \geq p_0$

$$n(2 - \epsilon)p \leq nJ(p) \leq J(pn) \leq 2pn$$

from which it follows that $(1 - \epsilon) \leq \frac{nJ(p)}{J(pn)} \leq 1$ and hence

$$(1 - \epsilon) \leq \frac{nJ(p) + 1}{J(pn) + 1} \leq 1$$

This completes the proof of the proposition. \square

We will now identify $K^{\text{gr}}(B)$

Lemma 2. *Let B be the algebra constructed above. Then $\mathbb{C}^{\text{gr}}(B) = \mathbb{C}(x)$.*

Proof. By definition, $K^{\text{gr}}(B) = \left\{ \frac{f}{g} \mid f, g \in B_n \text{ for some } n \right\}$. There is a natural morphism of fields

$$\phi : K^{\text{gr}}(B) \rightarrow \mathbb{C}(x)$$

²Which is a slightly stronger condition than required for approximability.

given by $\frac{P(x)/P_n(x)}{Q(x)/P_n(x)} \rightarrow P(x)/Q(x)$. Since $\phi\left(\frac{x/P_1(x)}{1/P_1(x)}\right) = x$, this map is surjective. As a morphism of fields, it is injective. This completes the proof of the lemma. \square

However, if we set $b_1 \in B_1 = 1$ then the set of valuations ν_{B,b_1} is infinite, since it contains ν_{z_i} for any choice of i .

But this now completes the proof of Theorem 1.2, since B is approximable, but since the set ν_{B,b_1} is infinite, B cannot be a graded subalgebra of the space of sections of a big line bundle.

REFERENCES

- [1] Boucksom, S. *Corps d'Okounkov (d'après Okounkov, Lazarsfeld-Mustata et Kaveh-Khovanskii)*. Séminaire Bourbaki no1059, Octobre 2012.
- [2] Chen, H. *Arithmetic Fujita Approximation*. Annales ENS, **43**, fasc. 4 (2010) 555-578.
- [3] di Biagio, L., Pacienza, G., *Restricted volumes of effective divisors*, Bull. Soc. Math. Fr. **144**, No. 2, 299-337 (2016).
- [4] Fujita, T.: *Approximating Zariski decomposition of big line bundles*. Kodai Math. J. 17(1), 1-3 (1994)
- [5] Kaveh, K.; Khovanskii, A.; *Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory*. Ann. of Math. (2) **176** (2012), no. 2, 925-978.
- [6] R. Lazarsfeld, M. Mustata, *Convex bodies associated to linear series*, Ann. Sci. Ecole Norm. Sup. (4) **42** (2009) 783-835.
- [7] A. Okounkov, *Why would multiplicities be log-concave?* in: The Orbit Method in Geometry and Physics, in: Progr. Math., vol. 213, 2003, pp. 329-347.